S620 - Introduction To Statistical Theory - Homework 2

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Assume squared error loss and let $Bias_\theta(d(X)) = E_\theta[d(X)] - \theta$. Show that:

$$risk = variance + (bias)^2$$

Proof: Let $\theta \in \Theta$. In symbols, we want to show that

$$R(\theta, d(X)) = Var_\theta d(X) + Bias_\theta(d(X))$$

Note that, by definition $Var_\theta d(X) = E_\theta[d(X)^2] - [E_\theta[d(X)]]^2$. Therefore,

$$\begin{aligned}
variance + (bias)^2 &= E_\theta[d(X)^2] - [E_\theta[d(X)]]^2 + (E_\theta[d(X)] - \theta)^2 && \text{by definitions} \\
&= E_\theta[d(X)^2] - [E_\theta[d(X)]]^2 + [E_\theta[d(X)]]^2 - 2\theta E_\theta[d(X)] + \theta^2 && \text{squaring} \\
&= E_\theta[d(X)^2] - 2\theta E_\theta[d(X)] + \theta^2 && \text{simplifying} \\
&= E_\theta[d(X)^2] - E_\theta[2\theta d(X)] + E_\theta[\theta^2] && \text{since } \theta \text{ is a constant with respect to } E_\theta \\
&= E_\theta[d(X)^2 - 2\theta d(X) + \theta^2] && \text{by linearity of } E_\theta \\
&= E_\theta[(\theta - d(X))^2] && \text{squaring} \\
&= E_\theta[L(\theta, d(X))] && \text{by definition of squared error loss} \\
&= R(\theta, d(X)) && \text{by definition of risk function} \\
&= risk
\end{aligned}$$

Thus, we obtain the result: $\boxed{risk = variance + (bias)^2}$